Z. Angew. Math. Phys. 61 (2010), 663–672

0044-2275/10/040663-10
*published online* January 12, 2010
DOI 10.1007/s00033-009-0050-y

**Zeitschrift für angewandte Mathematik und Physik ZAMP**

# Gauging magnetorotational instability

Isom Herron and Jeremy Goodman

**Abstract.** Previously (Z. Angew. Math. Phys. 57:615–622, 2006), we examined the axisymmetric stability of viscous resistive magnetized Couette flow with emphasis on flows that would be hydrodynamically stable according to Rayleigh's criterion: opposing gradients of angular velocity and specific angular momentum. A uniform axial magnetic field permeates the fluid. In this regime, magnetorotational instability (MRI) may occur. It was proved that MRI is suppressed, in fact no instability at all occurs, with insulating boundary conditions, when a term multipling the magnetic Prandtl number is neglected. Likewise, in the current work, including this term, when the magnetic resistivity is sufficiently large, MRI is suppressed. This shows conclusively that small magnetic dissipation is a feature of this instability for all magnetic Prandtl numbers. A criterion is provided for the onset of MRI.

**Mathematics Subject Classification (2000).** 76E25.

## 1. Introduction

Magnetorotational instability (MRI) is widely accepted as being important to theoretical astrophysics because it is the only linear instability known to grow significantly under the conditions prevailing in most accretion disks: an electrically conducting fluid; a positive gradient of specific angular momentum, $\partial(r^2\Omega)^2/\partial r > 0$; and a negative gradient of angular velocity, $\partial\Omega^2/\partial r < 0$ [1]. Earlier, but without this particular application in mind, Velikov [20] and Chandrasekhar [3] had examined the stability of a magnetically conducting fluid without dissipation, and had derived a counterpart to Rayleigh's criterion for this regime. Thus, the occurrence of an instability for a sufficiently large imposed axial magnetic field is to be expected.

Prior to that time, the importance of the Taylor–Couette instability had caused MRI to be overlooked. In this regime, as first shown by Chandrasekhar [2], the equations of motion can be scaled so that terms proportional to the magnetic Prandtl number $P_\mathrm{m} \equiv \nu/\eta \sim 10^{-6}$ are manifestly negligible, permitting a reduction of the axisymmetric stability analysis from tenth to eighth order in radial derivatives. Chandrasekhar's "small $P_\mathrm{m}$" approximation governed essentially all analyses of magnetized Couette flow for the following forty years; none of these works predicted MRI, nor did contemporary experiments observe it. Not long ago Goodman and Ji ([6], henceforth (GJ)), argued that it is natural to describe the onset of MRI in terms of dimensionless parameters that do not involve the viscosity $\nu$. It remained interesting to understand why the standard small-$P_\mathrm{m}$ approximation is inappropriate given that $P_\mathrm{m}$ is so very small in experiments. An attempt to clarify this was made by them, who argued that MRI cannot occur without one of the terms that Chandrasekhar dropped from his dimensionless equations on the grounds that it is proportional to $P_\mathrm{m}$. GJ proved their assertion only in the limit of a narrow gap between the cylinders. Subsequently, Herron and Goodman [8] extended GJ's results to wide-gap Couette flows. That

This material is based upon work supported in part by the U. S. Department of Energy under Grant No. DE-FG02-05ER25666 (to I. Herron) and by NSF award PHY-0821899 "Center for Magnetic Self-Organization in Laboratory and Astrophysical Plasmas" (to J. Goodman).

is, they proved that Chandrasekhar's reduced system of equations predict stability when $\partial(r^2\Omega)^2/\partial r > 0$, for insulating magnetic boundary conditions. The purpose in the current discussion is to examine criteria by which one can be sure that MRI is suppressed, even when the term proportional to $P_{\rm m}$ is retained, that is for the full linearized set of axisymmetric perturbation equations. It also provides a necessary criterion for MRI to occur, for which it may be gauged.

The present paper, like [3,6], and [8], concerns axisymmetric stability of background states with purely axial magnetic fields. Here we summarize what is known about stability of Couette flow with other magnetic geometries. All of these basic states are axisymmetric, with rotation profiles satisfying the conditions of the first paragraph above and with current-free fields. Herron and Soliman [9] have shown that an axisymmetric toroidal field is axisymmetrically stable; however, Rüdiger et al. [15] have found nonaxisymmetric instabilities. Current-free combinations of axial *and* toroidal fields are particularly interesting. Hollenbach and Rüdiger [10] have discovered an MRI-like instability of Couette flows with such fields that extends to very low magnetic Reynolds numbers, and which has since come to be known as "helical" MRI (HMRI). Notably, HMRI persists in the small-$P_{\rm m}$ approximation, when the terms shown by [6] and [8] to be necessary to axial MRI are neglected. This has enabled HMRI to be exhibited experimentally at low cylinder rotation speeds [17]. Opinions differ as to whether an absolute instability has been observed, rather than a convective one excited by boundary layers at the ends of the cylinders [11,12,18]. Although both convective and absolute HMRI instability exist on the stable side of the Rayleigh line, the range of the latter is more restricted [13], and that of the former is sensitive to the magnetic boundary conditions—i.e., wider for partially conducting conditions at one cylinder than for insulating conditions at both [13]. A satisfactory physical explanation of these trends has yet to be given.

In the next section, the governing equations are described, an abstract formulation is provided and then criteria for stability and instability are derived.

## 2. The governing equations and a criterion for stability

### 2.1. The governing equations

The basic flow is $\mathbf{v} = r\Omega\mathbf{e}_\theta$ (in cylindrical coordinates $r, \theta, z$) between two cylinders of radii $r_1, r_2$ and angular velocities $\Omega_1, \Omega_2$:

$$\Omega(r) = a + \frac{b}{r^2}, \quad a = \frac{\Omega_2 r_2^2 - \Omega_1 r_1^2}{r_2^2 - r_1^2}, \quad b = \frac{\Omega_1 - \Omega_2}{r_1^{-2} - r_2^{-2}}. \tag{2.1}$$

A uniform magnetic field $\mathbf{B} = B_0\mathbf{e}_z$ permeates the fluid.

The assumption that $\partial(r^2\Omega)^2/\partial r > 0$ implies $ab > 0$, and we take $a, b > 0$, so that $\Omega(r) = a + br^{-2} > 0$, without loss of generality. This profile supports a radially constant viscous torque $4\pi\rho\nu b$ per unit height $dz$, in which $\rho$ is the density of the fluid and $\nu$ the kinematic viscosity.

Following [4], linear perturbations are taken to be axisymmetric and sinusoidal in $z$. In the conventions of GJ [6],

$$\begin{aligned} \delta v_r &= \varphi_r(r,t)\sin kz, & \delta B_r/\sqrt{\mu\rho} &= \beta_r(r,t)\cos kz, \\ \delta v_\theta &= \varphi_\theta(r,t)\sin kz, & \delta B_\theta/\sqrt{\mu\rho} &= \beta_\theta(r,t)\cos kz, \\ \delta v_z &= \varphi_z(r,t)\cos kz, & \delta B_z/\sqrt{\mu\rho} &= \beta_z(r,t)\sin kz. \end{aligned}$$

The magnetic components have been scaled so as to have dimensions of Alfvén velocity; it is convenient to express the background field similarly, $V_A \equiv B_0/\sqrt{\mu\rho}$. The linearized equations of motion become (GJ)

$$\dot{\beta}_\theta = \eta(DD_* - k^2)\beta_\theta + kV_A\varphi_\theta + r\Omega'\beta_r, \tag{2.2}$$

$$\dot{\varphi}_\theta = \nu(DD_* - k^2)\varphi_\theta - kV_A\beta_\theta - r^{-1}(r^2\Omega)'\varphi_r, \tag{2.3}$$

$$\dot{\beta}_r = \eta(DD_* - k^2)\beta_r + kV_A\varphi_r, \tag{2.4}$$

$$(DD_* - k^2)\dot{\varphi}_r = \nu(DD_* - k^2)^2\varphi_r - kV_A(DD_* - k^2)\beta_r - 2\Omega k^2\varphi_\theta. \tag{2.5}$$

Primes denote radial derivatives of background quantities. For perturbations, we follow Chandrasekhar's notation $Df \equiv \partial f/\partial r$, $D_*f \equiv r^{-1}D(rf)$, and use dots for time derivatives. The vertical components $\varphi_z$ and $\beta_z$ have been eliminated from Eqs. (2.2)–(2.5) using $\nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{B} = 0$.

The boundaries are impenetrable and "no-slip", so that

$$\varphi_r = \varphi_r' = 0, \tag{2.6}$$

$$\varphi_\theta = 0, \quad \text{at } r = r_1, r_2. \tag{2.7}$$

The condition on $\varphi_r'$ derives from the continuity equation $D_*\varphi_r = -k\varphi_z$ since $\varphi_z = 0$. We take the cylinders to be perfectly insulating, and the magnetic perturbations to match onto exterior solutions of $\nabla \times \delta\mathbf{B} = 0$ that are well-behaved as $r \to 0$ and as $r \to \infty$ [14]:

$$\frac{\partial}{\partial r}(r\beta_r) = \beta_r \frac{[krI_0(kr)]}{I_1(kr)} \quad \text{at } r = r_1, \tag{2.8}$$

$$\frac{\partial}{\partial r}(r\beta_r) = -\beta_r \frac{[krK_0(kr)]}{K_1(kr)} \quad \text{at } r = r_2, \tag{2.9}$$

$$\beta_\theta = 0, \quad \text{at } r = r_1, r_2, \tag{2.10}$$

where $I_n(kr)$ and $K_n(kr)$ are the modified Bessel functions (of orders $n = 0, 1$ in this work). Eqs. (2.6)–(2.10) impose ten boundary conditions on the tenth-order differential system (2.2)–(2.5).

To proceed with the analysis, we assume a mode with a growth rate $s$, that is, disturbances are proportional to $e^{st}$. Equations (2.2 )–(2.5) become, with $\omega_A \equiv kV_A$,

$$s\beta_\theta = \eta(DD_* - k^2)\beta_\theta + \omega_A\varphi_\theta + r\Omega'\beta_r, \tag{2.11}$$

$$s\varphi_\theta = \nu(DD_* - k^2)\varphi_\theta - \omega_A\beta_\theta - r^{-1}(r^2\Omega)'\varphi_r, \tag{2.12}$$

$$s\beta_r = \eta(DD_* - k^2)\beta_r + \omega_A\varphi_r, \tag{2.13}$$

$$s(DD_* - k^2)\varphi_r = \nu(DD_* - k^2)^2\varphi_r - \omega_A(DD_* - k^2)\beta_r - 2\Omega k^2\varphi_\theta. \tag{2.14}$$

This is the full set of linearized axisymmetric instability equations, solved numerically by GJ [6]. Next we write this system in matrix operator form, in order to derive criteria for stability.

### 2.2. Abstract formulation

To simplify the proof, we make an abstract formulation. An operator notation is introduced, which clarifies the nature of the analysis. The system thereby becomes

$$-sM\varphi_r = \nu M^*M\varphi_r + \omega_A M_1\beta_r - 2\Omega k^2\varphi_\theta. \tag{2.15}$$

$$s\varphi_\theta = -\nu M_0\varphi_\theta - \omega_A\beta_\theta - r^{-1}(r^2\Omega)'\varphi_r, \tag{2.16}$$

$$s\beta_r = -\eta M_1\beta_r + \omega_A\varphi_r, \tag{2.17}$$

$$s\beta_\theta = -\eta M_0\beta_\theta + \omega_A\varphi_\theta + r\Omega'\beta_r. \tag{2.18}$$

In this notation, $M, M^*, M_0$, and $M_1$ all denote $-DD_* + k^2$, but are considered different operators because of the distinct boundary conditions satisfied by the functions on which they act, while $M^*M$ denotes $(-DD_* + k^2)^2$ [7]. That is, $M$ acts on functions that have the same boundary conditions as $\varphi_r$ (Eq. 2.6), $M^*$ assumes that the functions satisfy no particular boundary condition, whereas $M_0$ uses the boundary conditions of $\varphi_\theta$ and $\beta_\theta$ (Eqs. 2.7 and 2.10) and $M_1$ those of $\beta_r$ (Eqs. 2.8 and 2.9).

Introduce an inner product,

$$\langle f, g\rangle = \int_{r_1}^{r_2} r f(r)\bar{g}(r)\mathrm{d}r, \tag{2.19}$$

in which the overbar denotes complex conjugation. The differential operators $M, M^*M, M_0$, and $M_1$ all have the property of being *positive definite* in this inner product , so with integration by parts [8]

$$\langle M\varphi_r, \varphi_r\rangle > 0. \tag{2.20}$$

Likewise, it may be shown that $\langle M^*M\varphi_r, \varphi_r\rangle = \langle M\varphi_r, M\varphi_r\rangle \equiv \|M\varphi_r\|^2 > 0$. Also, for $\langle M_1\beta_r, \beta_r\rangle$ one notes that since $\beta_r$ satisfies (2.8) and (2.9), the boundary terms do not vanish after integration by parts; instead [8]

$$\langle M_1\beta_r, \beta_r\rangle = \int_{r_1}^{r_2} \left(r\left|D_*\beta_r\right|^2 + k^2 r\left|\beta_r\right|^2\right)\mathrm{d}r + \frac{kK_0(kr_2)}{K_1(kr_2)} r_2\left|\beta_r(r_2)\right|^2 + \frac{kI_0(kr_1)}{I_1(kr_1)} r_1\left|\beta_r(r_1)\right|^2 > 0.$$

We turn next to the criteria for stability and instability.

### 2.3. Criteria for stability and instability

**Theorem 2.1.** *MRI is suppressed, in fact no instability at all occurs with insulating boundary conditions, at those wave numbers for which*

$$\eta^2 k^4 > \frac{(\Omega_1 - a)^2\,\Omega_1}{a} > 0.$$

*Proof.* Write the set of Eqs. (2.15)–(2.18) as a system in matrix notation as

$$\mathcal{L}\mathbf{\Phi} = -s\mathcal{M}\mathbf{\Phi}, \tag{2.21}$$

where

$$\mathcal{L} = \begin{pmatrix} \nu M^*M & -2\Omega k^2 & \omega_A M_1 & 0 \\ 2\Omega k^2 & \nu a^{-1}\Omega k^2 M_0 & 0 & a^{-1}\Omega k^2\omega_A \\ -\omega_A & 0 & \eta M_1 & 0 \\ 0 & -a^{-1}\Omega k^2\omega_A & -a^{-1}\Omega k^2\left(r\Omega'\right) & \eta a^{-1}\Omega k^2 M_0 \end{pmatrix},$$

$$\mathbf{\Phi} = \begin{pmatrix} \varphi_r \\ \varphi_\theta \\ \beta_r \\ \beta_\theta \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} M & 0 & 0 & 0 \\ 0 & a^{-1}\Omega k^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a^{-1}\Omega k^2 \end{pmatrix}$$

and note that $r^{-1}(r^2\Omega)' = 2a$ is a positive constant. In a suitable inner product, form that of (2.21) with $\mathbf{\Phi}$ to obtain

$$(\mathcal{L}\mathbf{\Phi}, \mathbf{\Phi}) = -s\,(\mathcal{M}\mathbf{\Phi}, \mathbf{\Phi}),$$

making use of the inner product (2.19) as in (2.20). So, each of (2.15) to (2.18) produces a corresponding equation of functionals:

$$\langle \nu M^*M\varphi_r, \varphi_r\rangle - \langle 2\Omega k^2\varphi_\theta, \varphi_r\rangle + \langle \omega_A M_1\beta_r, \varphi_r\rangle = -s\langle M\varphi_r, \varphi_r\rangle, \tag{2.22}$$

$$\langle 2\Omega k^2\varphi_r, \varphi_\theta\rangle + \langle a^{-1}\Omega k^2\nu M_0\varphi_\theta, \varphi_\theta\rangle + \langle a^{-1}\Omega k^2\omega_A\beta_\theta, \varphi_\theta\rangle = -s\langle a^{-1}\Omega k^2\varphi_\theta, \varphi_\theta\rangle, \tag{2.23}$$

$$-\langle \omega_A\varphi_r, M_1\beta_r\rangle + \eta\langle M_1\beta_r, M_1\beta_r\rangle = -s\langle \beta_r, M_1\beta_r\rangle, \tag{2.24}$$

and

$$-\langle a^{-1}\Omega k^2\omega_A\varphi_\theta, \beta_\theta\rangle + \langle a^{-1}\Omega k^2\left(-r\Omega'\right)\beta_r, \beta_\theta\rangle + \langle a^{-1}\Omega k^2\eta M_0\beta_\theta, \beta_\theta\rangle = -s\langle a^{-1}\Omega k^2\beta_\theta, \beta_\theta\rangle. \tag{2.25}$$

Adding (2.22)–(2.25) and taking real parts gives

$$\nu \left\|M\varphi_r\right\|^2+\nu \operatorname{Re}\langle a^{-1}\Omega k^2 M_0\varphi_\theta,\varphi_\theta\rangle+\eta\langle M_1\beta_r,M_1\beta_r\rangle-\operatorname{Re}\langle a^{-1}\Omega k^2\left(r\Omega'\right)\beta_r,\beta_\theta\rangle+\operatorname{Re}\langle a^{-1}\Omega k^2\eta M_0\beta_\theta,\beta_\theta\rangle$$
$$=-\operatorname{Re}(s)\left(\langle M\varphi_r,\varphi_r\rangle+\langle a^{-1}\Omega k^2\varphi_\theta,\varphi_\theta\rangle+\langle M_1\beta_r,\beta_r\rangle+\langle a^{-1}\Omega k^2\beta_\theta,\beta_\theta\rangle\right). \tag{2.26}$$

It was established by Synge [19] and by Chandrasekhar [4], that for the azimuthal velocity function $\varphi_\theta$, the following is true:

$$\operatorname{Re}\left\langle \Omega(-DD_*+k^2)\varphi_\theta,\varphi_\theta\right\rangle>0, \tag{2.27}$$

by virtue of (2.7). Likewise, for the azimuthal component of the magnetic field $\beta_\theta$,

$$\operatorname{Re}\langle\Omega(-DD_*+k^2)\beta_\theta,\beta_\theta\rangle>0, \tag{2.28}$$

using (2.10). The first two terms on the left side of (2.26) are manifestly stabilizing but vanish in the inviscid limit. Hence we are led to consider the last three terms on the left side (2.26) involving the components of the magnetic field. We observe that

$$\langle M_1\beta_r,M_1\beta_r\rangle=\langle(-DD_*+k^2)\beta_r,(-DD_*+k^2)\beta_r\rangle\geq k^4\langle\beta_r,\beta_r\rangle.$$

With a little more calculation it may shown that

$$\operatorname{Re}\langle\Omega(-DD_*+k^2)\beta_\theta,\beta_\theta\rangle\geq k^2\langle\Omega\beta_\theta,\beta_\theta\rangle. \tag{2.29}$$

Hence, comparing these the result is

$$\eta\langle M_1\beta_r,M_1\beta_r\rangle-\operatorname{Re}\langle a^{-1}\Omega k^2\left(r\Omega'\right)\beta_r,\beta_\theta\rangle+\operatorname{Re}\langle a^{-1}\Omega k^2\eta M_0\beta_\theta,\beta_\theta\rangle$$
$$\geq\eta k^4\left\|\beta_r\right\|^2-a^{-1}k^2\operatorname{Re}\langle r\Omega'\Omega\beta_r,\beta_\theta\rangle+a^{-1}\eta k^4\langle\Omega\beta_\theta,\beta_\theta\rangle. \tag{2.30}$$

As a quadratic form, (2.30) will be non-negative as long as

$$\int_{r_1}^{r_2}\left[\eta k^4\left|\beta_r\right|^2-a^{-1}k^2 r\Omega'\Omega\operatorname{Re}\left(\beta_r\bar{\beta}_\theta\right)+a^{-1}\eta k^4\Omega\left|\beta_\theta\right|^2\right]r\,\mathrm{d}r\geq 0. \tag{2.31}$$

The integrand is positive if

$$\eta^2k^4>\frac{\left(r\Omega'\right)^2\Omega}{4a}>0, \tag{2.32}$$

everywhere in the interval $r_1<r<r_2$. This will be true when the left side exceeds the value where the center expression has its maximum, at $r=r_1$, because $\Omega$ and $(r\Omega')^2$ decrease with $r$. We have that $\Omega(r_1)=\Omega_1$ and $(r\Omega')^2|_{r=r_1}=4b^2/r_1^4=4(\Omega_1-a)^2$. The result is

$$\eta^2k^4>\frac{\left(\Omega_1-a\right)^2\Omega_1}{a}>0.$$

An immediate consequence of this from (2.26) is $\operatorname{Re}(s)<0$, and hence stability. Since $\operatorname{Re}(s)$ *is* strictly negative, there are no marginal modes. □

**Theorem 2.2.** *The stability condition may be strengthened to: No instability at all occurs with insulating boundary conditions, for which*

$$\frac{\pi^2\left(\omega_{\min}^2+1\right)\eta^2}{r_2^4\ln^2(r_2/r_1)}\geq\frac{\left(r\Omega'\right)^2\Omega}{8a}$$

*throughout the flow, where* $\omega_{\min}$ *is the smallest positive solution of the equation*

$$\omega\tan\left[\omega\ln(r_2/r_1)\right]=1.$$

*Proof.* First an extension of (2.29) is established as

$$\mathrm{Re}\langle \Omega(-DD_* + k^2)\beta_\theta, \beta_\theta\rangle \geq \lambda_{\min}\langle \Omega\beta_\theta, \beta_\theta\rangle, \tag{2.33}$$

where $\lambda_{\min}$ is the minimum eigenvalue of $-D_*D + k^2$ with the boundary conditions (2.9). Thus since

$$-DD_* = -D_*D + \frac{1}{r^2}, \tag{2.34}$$

we have

$$\mathrm{Re}\langle \Omega(-DD_* + k^2)\beta_\theta, \beta_\theta\rangle \geq \mathrm{Re}\langle \Omega(-D_*D + k^2)\beta_\theta, \beta_\theta\rangle.$$

Next, since $\Omega = a + br^{-2}$, $a, b \geq 0$, it is sufficient to show that $\mathrm{Re}\langle r^{-2}(-DD_* + k^2)\beta_\theta, \beta_\theta\rangle \geq \lambda_{\min}\langle r^{-2}\beta_\theta, \beta_\theta\rangle$:
Consider

$$\mathrm{Re}\langle r^{-2}(-DD_* + k^2)\beta_\theta, \beta_\theta\rangle = \mathrm{Re}\int_{r_1}^{r_2} \frac{\bar{\beta}_\theta}{r}\left\{-\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\beta_\theta)\right] + k^2\beta_\theta\right\}\mathrm{d}r$$

(integrate by parts)

$$= \mathrm{Re}\int_{r_1}^{r_2} \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\bar{\beta}_\theta}{r}\right)\left\{\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^2(r^{-1}\beta_\theta)\right]\right\}\mathrm{d}r + \int_{r_1}^{r_2} k^2\left|\beta_\theta\right|^2 r^{-1}\mathrm{d}r$$

$$= \int_{r_1}^{r_2}\left|\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\beta_\theta}{r}\right)\right|^2 r\,\mathrm{d}r + \underbrace{\int_{r_1}^{r_2}\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\bar{\beta}_\theta}{r}\right)\frac{\beta_\theta}{r} + \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\beta_\theta}{r}\right)\frac{\bar{\beta}_\theta}{r}\right]\mathrm{d}r}_{\text{a vanishing total derivative}}$$

$$+ \int_{r_1}^{r_2} k^2\left|\beta_\theta\right|^2 r^{-1}\mathrm{d}r = \langle D(\beta_\theta/r), D(\beta_\theta/r)\rangle + k^2\langle \beta_\theta/r, \beta_\theta/r\rangle > 0.$$

Put $f = \beta_\theta/r$. The estimate

$$\lambda = \min\frac{\langle Df, Df\rangle + k^2\langle f, f\rangle}{\langle f, f\rangle}$$

leads to the eigenvalue problem [5, Chapter 4]

$$\left(-D_*D + k^2\right)f = \lambda f,$$
$$f(r_1) = f(r_2) = 0,$$

which produces the desired minimum. This establishes (2.33).

Next, let $q^2_{\min}$ be the smallest eigenvalue of the operator $M_1$, so that

$$\langle M_1\beta_r, \beta_r\rangle = \langle(-DD_* + k^2)\beta_r, \beta_r\rangle \geq q^2_{\min}\langle\beta_r, \beta_r\rangle, \tag{2.35}$$

and in (2.30)

$$\langle M_1\beta_r, M_1\beta_r\rangle = \left\|M_1\beta_r\right\|^2 \geq \langle M_1\beta_r, \beta_r\rangle^2/\left\|\beta_r\right\|^2. \tag{2.36}$$

Again, from (2.34), we have

$$\langle(-DD_* + k^2)\beta_r, \beta_r\rangle \geq \langle(-D_*D + k^2)\beta_r, \beta_r\rangle, \tag{2.37}$$

so $q_{\min}^2$ will be shown to be bounded below by the smallest eigenvalue of the operator $-D_*D + k^2$, with the same boundary conditions as $M_1$. To construct a lower bound that does not depend on Bessel functions, note that the insulating boundary conditions for $\beta_r$ are equivalent to (with $k \geq 0$)

$$\frac{rD\beta_r}{\beta_r} = \begin{cases} krI_1'(kr)/I_1(kr) \geq 1 & \text{at } r = r_1, \\ krK_1'(kr)/K_1(kr) \leq -kr & \text{at } r = r_2. \end{cases} \tag{2.38}$$

The first inequality in (2.38) follows from the fact that all the terms of the power series in $x$ for $I_1(x)$ are nonnegative, while the second follows from the integral representation

$$K_1(x) = \int_0^\infty e^{-x\cosh t} \cosh t \, \mathrm{d}t \quad (x > 0).$$

Owing to these inequalities, replacing the boundary conditions (2.38) with

$$\frac{\mathrm{d}\ln\beta_r}{\mathrm{d}\ln r} = \begin{cases} 1 & \text{at } r = r_1, \\ 0 & \text{at } r = r_2, \end{cases} \tag{2.39}$$

can only decrease the minimum eigenvalue [5, Chapter 2]. Set $x = \ln r$, so that the boundary conditions become

$$\frac{\mathrm{d}\beta_r}{\mathrm{d}x} = \begin{cases} \beta_r & \text{at } x = x_1, \\ 0 & \text{at } x = x_2. \end{cases} \tag{2.40}$$

The functional transforms as

$$\begin{aligned} \langle(-D_*D)\beta_r, \beta_r\rangle &= -\int_{r_1}^{r_2} \bar{\beta}_r \frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\beta_r}{\mathrm{d}r}\right)\mathrm{d}r \\ &= -\int_{x_1}^{x_2} \bar{\beta}_r \frac{\mathrm{d}^2\beta_r}{\mathrm{d}x^2}\mathrm{d}x = \int_{x_1}^{x_2} \left|\frac{\mathrm{d}\beta_r}{\mathrm{d}x}\right|^2 \mathrm{d}x + |\beta_r(x_1)|^2 > 0. \end{aligned}$$

Let $\omega_{\min}^2$ be given by the estimate

$$\omega_{\min}^2 = \min \frac{\int_{x_1}^{x_2} \left|\frac{\mathrm{d}\beta_r}{\mathrm{d}x}\right|^2 \mathrm{d}x + |\beta_r(x_1)|^2}{\int_{x_1}^{x_2} |\beta_r|^2 \, \mathrm{d}x},$$

where the minimum is taken over functions satisfying (2.40) [5, Chapter 4]. Thus, we are led to consider the eigenfunctions of $-\mathrm{d}^2/\mathrm{d}x^2$ with the boundary conditions (2.40), which are of the form $\beta_r = A\cos(\omega x) + B\sin(\omega x)$, with $\omega$ satisfying

$$\omega \tan[\omega(x_2 - x_1)] = 1. \tag{2.41}$$

Then, because $\int_{x_1}^{x_2} |\beta_r|^2 \mathrm{d}x = \langle r^{-2}\beta_r, \beta_r\rangle \geq r_2^{-2}\langle\beta_r, \beta_r\rangle$, it follows by using (2.34) and (2.37), that

$$\langle M_1\beta_r, \beta_r\rangle \geq \left(k^2 + \frac{\omega_{\min}^2 + 1}{r_2^2}\right)\langle\beta_r, \beta_r\rangle. \tag{2.42}$$

So from (2.36) we may conclude that

$$\langle M_1\beta_r, M_1\beta_r\rangle \geq \left(k^2 + \frac{\omega_{\min}^2 + 1}{r_2^2}\right)^2\langle\beta_r, \beta_r\rangle. \tag{2.43}$$

Employing (2.41), (2.42) may be further refined. Using the fact that

$$\tan x \leq x/\left[1 - (2x/\pi)^2\right] \quad \text{for } 0 \leq x < \pi/2,$$

one has

$$\omega_{\min}^2 \geq \left[\ln\frac{r_2}{r_1} + \left(\frac{2}{\pi}\ln\frac{r_2}{r_1}\right)^2\right]^{-1}. \tag{2.44}$$

A lower bound on $\lambda_{\min}$ can be established in a similar way. That is, for functions $f(r)$ having the boundary conditions of $\beta_\theta$ and $\varphi_\theta$,

$$\begin{aligned}\langle(-D_* D)f, f\rangle &= -\int_{x_1}^{x_2} \bar{f}\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\mathrm{d}x \\ &\geq \frac{\pi^2}{(x_2-x_1)^2}\int_{x_1}^{x_2}|f|^2\,\mathrm{d}x = \frac{\pi^2}{(x_2-x_1)^2}\int_{r_1}^{r_2} r^{-1}|f|^2\,\mathrm{d}r \\ &= \frac{\pi^2}{(x_2-x_1)^2}\langle r^{-2}f, f\rangle \geq \frac{\pi^2 r_2^{-2}}{\ln^2(r_2/r_1)}\langle f, f\rangle;\end{aligned}$$

whence

$$\lambda_{\min} \geq k^2 + \left(\frac{\pi}{r_2\ln(r_2/r_1)}\right)^2 \tag{2.45}$$

With (2.43) and (2.45) the counterpart of (2.30) is now

$$\begin{aligned}&\eta\langle M_1\beta_r, M_1\beta_r\rangle - \mathrm{Re}\langle a^{-1}\Omega k^2\,(r\Omega')\,\beta_r, \beta_\theta\rangle + \mathrm{Re}\langle a^{-1}\Omega k^2\eta M_0\beta_\theta, \beta_\theta\rangle \\ &\geq \eta\left(k^2 + \frac{\omega_{\min}^2+1}{r_2^2}\right)^2\langle\beta_r, \beta_r\rangle - a^{-1}k^2\,\mathrm{Re}\langle r\Omega'\Omega\beta_r, \beta_\theta\rangle \\ &\quad + a^{-1}\eta k^2\left[k^2 + \left(\frac{\pi}{r_2\ln(r_2/r_1)}\right)^2\right]\langle\Omega\beta_\theta, \beta_\theta\rangle.\end{aligned} \tag{2.46}$$

Observing just the terms of order $k^2$ on the extreme right side of (2.46), we see that this inequality is positive, by the same reasoning as for (2.31), as long as

$$\frac{\pi^2\left(\omega_{\min}^2+1\right)\eta^2}{r_2^4\ln^2(r_2/r_1)} \geq \frac{(r\Omega')^2\,\Omega}{8a} \tag{2.47}$$

throughout the flow. □

We have the following corollary which provides a gauge to the onset of MRI.

**Corollary 2.3.** *A necessary condition for the occurrence of MRI to axisymmetric disturbances with insulating boundary conditions is*

$$\frac{(r\Omega')^2\,\Omega}{8a} > \frac{\pi^2\left(\omega_{\min}^2+1\right)\eta^2}{r_2^4\ln^2(r_2/r_1)} \tag{2.48}$$

*somewhere in the flow.*

## 3. Concluding comments

We have derived two inequalities which determine conditions under which stability is predicted, so that MRI will not occur. Clearly the analytical results are not sharp but do provide rigorous bounds and regions in which unstable modes will not occur. As a consequence, the only terms which may contribute to MRI are identified and bounded.

Inequality (2.44) becomes an equality in the limits $r_2/r_1 \to 1$ and $r_2/r_1 \to \infty$. This shows that the lower bound (2.42) on the eigenvalue of $M_1$, and hence on the magnetic diffusion rate, is dominated by the width of the gap rather than by $k^2$ if $k \ll \pi/2(r_2 - r_1)$. Also, inequality (2.32) is recovered from (2.46) when $r_2 \to \infty$, for $r_1$ fixed. The stronger of these results is (2.47), since it holds for all wave numbers $k$. This condition reflects the fact that there must be a sufficiently weak magnetic diffusivity for instability. It shows conclusively that small magnetic dissipation is a feature of this instability for all magnetic Prandtl numbers as shown by (2.48).

Our analysis has been for insulating boundary conditions and the question naturally arises how might conducting conditions affect the outcome. This will result in a change in the boundary conditions on both $\beta_r$ and $\beta_\theta$, to be sure. First, in the case of $\beta_r$, the conducting boundary conditions become the same as (2.10), which renders the analysis from (2.35) down to (2.44) unnecessary. Still needing an estimate, the smallest eigenvalue of the operator $M_0$ is used instead. Next, affecting the outcome more profoundly, the conducting conditions are $D_*\beta_\theta = 0$ on the walls for the azimuthal perturbation. So the estimate (2.33) does not immediately follow. However, should the outer cylinder have a conducting boundary condition, with inner insulating, then a similar result may be derived. The analysis below (2.34) can be modified appropriately, where the term described as having "a vanishing total derivative" only vanishes at $r = r_1$. The part from $r = r_2$ contributes positively to the functional. Both cylinders conducting is worthy of a numerical study as has been carried out elsewhere [6,16].

## References


1. Balbus, S.A., Hawley, J.F.: Instability, turbulence, and enhanced transport in accretion disks. Rev. Mod. Phys. **70**, 1–53 (1998)
2. Chandrasekhar, S.: The stability of viscous flow between rotating cylinders in the presence of a magnetic field. Proc. R. Soc. Lond. A **216**, 293–309 (1953)
3. Chandrasekhar, S.: The stability of non-dissipative Couette flow in hydromagnetics. Proc. Natl. Acad. Sci. USA **46**, 253–257 (1960)
4. Chandrasekhar, S.: Hydrodynamic and Hydromagnetic Stability. Oxford University Press, London (1961)
5. Cochran, J.A.: Applied Mathematics: Principles, Techniques, and Applications. Wadsworth, Belmont (1982)
6. Goodman, J., Ji, H.: Magnetorotational instability of dissipative Couette flow. J. Fluid Mech. **462**, 365–382 (2002)
7. Herron, I.H.: Onset of instability in hydromagnetic Couette flow. Anal. Appl. **2**, 145–159 (2004)
8. Herron, I.H., Goodman, J.: The small Prandtl number approximation suppresses magnetorotational instability. J. Appl. Math. Phys. (Z. Angew. Math. Phys.) **57**, 615–622 (2006)
9. Herron, I., Soliman, F.: The stability of Couette flow in a toroidal magnetic field. Appl. Math. Lett. **19**, 1113–1117 (2006)
10. Hollenbach, R., Rüdiger, G.: New type of magnetorotational instability in cylindrical Taylor–Couette flow. Phys. Rev. Lett. **95**, 124501 (2005)
11. Liu, W., Goodman, J., Herron, I., Ji, H.: Helical magnetorotational instability in magnetized Taylor–Couette flow. Phys. Rev. E **74**, 056302 (2006)
12. Liu, W.: Noise-sustained convective instability in a magnetized Taylor–Couette flow. Astrophys. J. **692**, 998–1003 (2009)
13. Priede, J., Gerbeth, G.: Absolute versus convective helical magnetorotational instability in a Taylor–Couette flow. Phys. Rev. E **79**, 046310 (2009)
14. Roberts, P.H.: The stability of hydromagnetic Couette flow. Proc. Camb. Phil. Soc. **60**, 635–651 (1964)
15. Rüdiger, G., Hollerbach, R., Schultz, M., Elstner, D.: Destabilization of hydrodynamically stable rotation laws by azimuthal magnetic fields. Mon. Notices R. Astron. Soc. **377**, 1481–1487 (2007)
16. Rüdiger, G., Hollerbach, R., Schultz, M., Shalybkov, D.A.: The stability of MHD Taylor–Couette flow with current-free spiral magnetic fields between conducting cylinders. Astron. Nachr. **326**, 409–413 (2005)
17. Stefani, F., Gundrum, T., Gerbeth, G., Rüdiger, G., Schultz, M., Szklarski, J., Hollerbach, R.: Experimental evidence for magnetorotational instability in a Taylor–Couette flow under the influence of a helical magnetic field. Phys. Rev. Lett. **97**, 184502 (2006)
18. Stefani, F., Gerbeth, G., Gundrum, T., Hollerbach, R., Priede, J., Rüdiger, G., Szklarski, J.: Helical magnetorotational instability in a Taylor–Couette flow with strongly reduced Ekman pumping. Phys. Rev. E (2009, submitted). `arXiv:0904.1027`
19. Synge, J.L.: On the stability of a viscous liquid between rotating coaxial cylinders. Proc. R. Soc. Lond. A **167**, 250–256 (1938)

20. Velikhov, E.P.: Stability of an ideally conducting liquid flowing between cylinders rotating in a magnetic field. Sov. Phys. JETP **36**, 995–998 (1959)

Isom Herron
Rensselaer Polytechnic Institute
Troy
NY 12180
USA
e-mail: herroi@rpi.edu

Jeremy Goodman
Princeton University Observatory
Princeton
NJ 08544
USA

(Received: July 15, 2009; revised: November 25, 2009)